Procedure for Graphing Polynomial Functions

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots\ldots + a_1 x + a_0$$

As an example, we will examine the following polynomial function:

$$P(x) = 2x^3 - 3x^2 - 23x + 12$$

To graph *P*(*x*):

1. **Determine the far-left and far-right behavior by examining the leading coefficient and degree of the polynomial.**

   The ***sign of the leading coefficient*** determines if the graph's far-right behavior. If the leading coefficient is *positive*, then the graph will be going up to the far right. If the leading coefficient is *negative*, then the graph will be going down to the far right.

   The ***degree of the polynomial*** determines the relationship between the far-left behavior and the far-right behavior of the graph. If the degree of the polynomial is *even*, then both ends of the graph go in the same direction; either both ends go up or both ends go down. If the degree of the polynomial is *odd*, then the ends of the graph go in opposite directions, one end up and one end down.

   Example: $P(x) = 2x^3 - 3x^2 - 23x + 12$

   The leading term in our polynomial is $2x^3$. The leading coefficient is +2 and the degree is 3. The leading coefficient is *positive*, and the degree is *odd*. When we combine these two pieces of information, we can conclude that this graph will be going **up on the far right and down to the far left.**

2. **Find the *y*-intercept**
   To find the *y*-intercept you will evaluate the function when $x = 0$, or $P(0)$

   Example: $P(x) = 2x^3 - 3x^2 - 23x + 12$

   $P(0) = 2(0)^3 - 3(0)^2 - 23(0) + 12 = 12$
   Therefore, **the *y*-intercept is (0, 12)**.

3. **Find the *x*-intercepts**

   Another way of saying "find the *x*-intercepts of a function," is to say "find the zeros of the function." To find the *x*-intercept you will evaluate the function when $y = 0$, or $P(x)=0$.

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Procedure for Finding Zeros of a Polynomial Function

a) Gather general information

Determine the degree of the polynomial (gives the most zeros possible)

Example: $P(x) = 2x^3 - 3x^2 - 23x + 12$

The degree is 3, **so this polynomial will have at most 3 zeros** (or 3 x-intercepts).

Apply *Descartes' Rule of Signs* - This rule will tell you the maximum number of positive real zeros and negative real zeros.

To find the number of positive real zeros, you must count the number of sign changes in $P(x)$. The number of positive real zeros is equal to either the number of sign changes or to the number of sign changes minus a multiple of 2.

Example: $P(x) = 2x^3 - 3x^2 - 23x + 12$

There are two sign changes in $P(x)$, so we can conclude that there are **2 or 0** $(2 - 2 = 0)$ **positive real zeros**.

To find the number of negative real zeros, you must count the number of sign changes in $P(-x)$. The number of negative real zeros is equal to either the number of sign changes or to the number of sign changes minus a multiple of 2.

Example: $P(x) = 2x^3 - 3x^2 - 23x + 12$

$$P(-x) = 2(-x)^3 - 3(-x)^2 - 23(-x) + 12$$
$$P(-x) = -2x^3 - 3x^2 + 23x + 12$$

There is one sign change in $P(-x)$, so we can conclude that there is **one negative real zero.**

List possible zeros – use the *Rational Zero Theorem*

The Rational Zero Theorem states that to find the possible zeros of a polynomial you find factors of the constant (p) divided by factors of the leading coefficient (q). These fractions will be the possible x-intercepts of the polynomial.

Example: $P(x) = 2x^3 - 3x^2 - 23x + 12$

$$\frac{p}{q} = \frac{factors\ of +12}{factors\ of +2} = \frac{\pm1, \pm2, \pm3, \pm4, \pm6, \pm12}{\pm1, \pm2} = \pm1, \pm\frac{1}{2}, \pm2, \pm3, \pm\frac{3}{2}, \pm4, \pm6, \pm12$$

Therefore, the **possible zeros of $P(x)$ are $\pm1, \pm\frac{1}{2}, \pm2, \pm3, \pm\frac{3}{2}, \pm4, \pm6, \pm12$.**

b) Check suspects

Use synthetic division to test the list you created above. Be aware of the *Upper and Lower bound* rules; these may eliminate some of your possibilities as you discover the bounds.

Upper Bound: to find the smallest positive-integer upper bound, use synthetic division with 1, 2, 3, … as test values. If the leading coefficient is positive, you will have the upper bound of your test values when the bottom row is all positive numbers. If the leading coefficient is negative, you will have the upper bound of the test values when the bottom row is all negative numbers. There will be no zeros greater than the upper bound value; therefore, no test value greater than this number will need to be tested by synthetic division.

Lower Bound: to find the largest negative-integer lower bound, use synthetic division with -1,-2,-3, … as test values. When the bottom row is alternating signs, this gives the lower bound of the test values. There will be no zeros less than the lower bound value; therefore, no test value less than this number will need to be tested by synthetic division.

Example: $P(x) = 2x^3 - 3x^2 - 23x + 12$

Test values: $\pm 1, \pm\frac{1}{2}, \pm 2, \pm 3, \pm\frac{3}{2}, \pm 4, \pm 6, \pm 12$

+1	2	-3	-23	12
		2	-1	-24
	2	-1	-24	-12

-1	2	-3	-23	12
		-2	12	11
	+2	-6	-11	22

We have alternating signs in the bottom row, so according to the *Lower Bound Theorem* we will not have any zeros less than -1.

+1/2	2	-3	-23	12
		1	-1	-12
	+2	-2	-24	0

The *Factor Theorem* states that a polynomial $P(x)$ has a factor $(x - c)$ if and only if $P(c) = 0$.

Since we have a remainder of 0, we know that $P(+½) = 0$. Therefore, $(x - ½)$ is a factor of $P(x)$.

We can now create a reduced polynomial using the results from our synthetic division.

$$\boldsymbol{P(x) = (x - ½)(2x^2 - 2x - 24)}$$

c) Work with reduced polynomial

If a reduced polynomial is of degree 2, find zeros by factoring or applying the quadratic formula.

If a reduced polynomial is of degree 3 or greater, repeat steps a-c of finding zeros.

Example: $P(x) = 2x^3 - 3x^2 - 23x + 12$

$$P(x) = (x - ½)(2x^2 - 2x - 24)$$

Since our reduced polynomial has a degree of 2, we can factor to get the remaining zeros.

$2x^2 - 2x - 24$
$(2x^2 + 6x) + (-8x - 24)$
$2x(x + 3) - 8(x + 3)$
$(2x - 8)(x + 3)$

Factors of -48	Sum of -2
+6, -8	-2

$P(x) = (x - ½)(2x - 8)(x + 3)$

Remember: to find the x-intercepts set P(x) = 0.

$0 = (x - ½)(2x - 8)(x + 3)$

$0 = x - ½$ $\quad 0 = 2x - 8$ $\quad 0 = x + 3$

$x = ½$ $\quad x = 4$ $\quad x = -3$

The *x*-intercepts are (½, 0), (4, 0), and (-3, 0).

4. **Find additional points** – you can find additional points by selecting any value for *x* and plugging the value into the equation and then solving for *y*

   It is most helpful to select values of *x* that fall in-between the zeros you found in step 3 above.

x	$P(x) = 2x^3 - 3x^2 - 23x + 12$	(x,y)
-5	$P(-5) = 2(-5)^3 - 3(-5)^2 - 23(-5) + 12 = -198$	**(-5, -198)**
-3/2	$P(-3/2) = 2(-3/2)^3 - 3(-3/2)^2 - 23(-3/2) + 12 = 33$	**(-3/2, 33)**
-1	$P(-1) = 2(-1)^3 - 3(-1)^2 - 23(-1) + 12 = 30$	**(-1, 30)**
2	$P(2) = 2(2)^3 - 3(2)^2 - 23(2) + 12 = -30$	**(2, -30)**
7/2	$P(7/2) = 2(7/2)^3 - 3(7/2)^2 - 23(7/2) + 12 = -19½$	**(7/2, -19½)**
5	$P(5) = 2(5)^3 - 3(5)^2 - 23(5) + 12 = 72$	**(5, 72)**

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5. **Check for symmetry** (check with respect to *x*-axis, *y*-axis, and origin)
   a. To check to see if a graph is symmetrical with respect to the *x*-axis, simply replace "*y*" with a "-*y*" and simplify. If $P(x) = -(P(x))$ than the graph is symmetrical with respect to the *x*-axis.

      Example: $P(x) = 2x^3 - 3x^2 - 23x + 12$
      $y = 2x^3 - 3x^2 - 23x + 12$
      $-y = 2x^3 - 3x^2 - 23x + 12$
      $y = -2x^3 + 3x^2 + 23x - 12$

      $P(x) \neq -(P(x))$; therefore, **the graph is not symmetrical with respect to the *x*-axis**.

   b. To check to see if a graph is symmetrical with respect to the *y*-axis, simply replace "*x*" with a "-*x*" and simplify. If $P(x) = P(-x)$ than the graph is symmetrical with respect to the *y*-axis.

      Example: $P(x) = 2x^3 - 3x^2 - 23x + 12$
      $P(-x) = 2(-x)^3 - 3(-x)^2 - 23(-x) + 12$
      $P(-x) = -2x^3 - 3x^2 + 23x + 12$

      $P(x) \neq P(-x)$ therefore **the graph is not symmetrical with respect to the y-axis.**

   c. To check if a graph is symmetrical with respect to the origin, simply replace both "*x*" and "*y*" with "-*x*" and "-*y*" and simply. If the equation remains the same, then the graph is symmetrical with respect to the origin.

      Example: $P(x) = 2x^3 - 3x^2 - 23x + 12$
      $y = 2x^3 - 3x^2 - 23x + 12$
      $-y = 2(-x)^3 - 3(-x)^2 - 23(-x) + 12$
      $y = 2x^3 + 3x^2 - 23x - 12$

      $P(x) \neq -(P(-x))$; therefore**, the graph is not symmetrical with respect to the origin**.

6. Sketch the graph

a. Plot the *x*- and *y*-intercepts

For our example these are: **(0, 12)**, **(½, 0), (4, 0), and (-3, 0)**

b. Plot the additional points you found

For our example these are: **(-5, -198), (-3/2, 33), (-1, 30), (2, -30), (7/2, -19½), (5, 72)**

c. Graph the curve