Higher Order Differential Equations

In advanced engineering and math courses, differential equations are used more frequently and are often more difficult than first order equations. This handout will discuss how to calculate and work with Wronskians and homogeneous equations as well as methods for solving higher order differential equations including reduction of order, undetermined coefficients, and variation of parameters. For information about Laplace Transforms, refer to the Academic Center for Excellence's Laplace Transforms handout.

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Using Wronskians to Verify Independent Solutions

Because higher order differential equations contain more than one solution, methods are required to verify that these solutions are unique and independent. The Wronskian method will dictate if a solution is unique. A Wronskian is the determinant of a matrix that is filled with an equation's solutions on the first row and the solutions' derivatives on the second row. The matrix format can be seen below.

$$W = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = f(x)g'(x) - g(x)f'(x)$$

If the Wronskian does not equal zero, $W \neq 0$, then the set of solutions is unique; however, if $W = 0$, then the set of solutions is not unique. The process for using Wronskians is given below:

Step 1: Find the derivative of both solutions.

Step 2: Place the solutions and derivatives in a matrix following the layout above.

Step 3: Solve for the Wronskian.

Note, if there are more than two solutions, repeat these steps with two solutions at a time. For example, if the problem has three solutions, $f(x)$, $g(x)$, and $h(x)$, find the Wronskian for $f(x)$ and $g(x)$, repeat for $f(x)$ and $h(x)$, then repeat with $g(x)$ and $h(x)$ in order to confirm that each solution is unique relative to the others.

Example 1: Determine if the following set of solutions is unique.

$$[y_1 = e^{8x}, y_2 = e^{-8x}]$$

Step 1: Find the derivative of both solutions:

$$y_1' = 8e^{8x} \qquad\qquad y_2' = -8e^{-8x}$$

Step 2: Place the solutions and derivatives in a matrix in the following layout:

$$W = \begin{vmatrix} e^{8x} & e^{-8x} \\ 8e^{8x} & -8e^{-8x} \end{vmatrix}$$

Step 3: Solve for the Wronskian:

$$W = (e^{8x})(-8e^{-8x}) - (e^{-8x})(8e^{8x})$$

$$W = -8e^{0x} - 8e^{0x}$$

$$W = -8 - 8 = -16 \neq 0$$

This set of solutions is unique because the Wronskian is not equal to zero.

Example 2: Determine if the following set of solutions is unique.

$$[y_1 = 7e^{4x}, y_2 = 6e^{4x+2}]$$

Step 1: Find the derivative of both solutions:

$$y_1' = 28e^{4x} \qquad\qquad y_2' = 24e^{4x+2}$$

Step 2: Plug the solutions and derivatives into a matrix:

$$\begin{vmatrix} 7e^{4x} & 6e^{4x+2} \\ 28e^{4x} & 24e^{4x+2} \end{vmatrix} = W$$

Step 3: Solve the Wronskian:

$$W = (7e^{4x})(24e^{4x+2}) - (6e^{4x+2})(28e^{4x})$$

$$W = 168e^{8x+2} - 168e^{8x+2}$$

$$W = 0$$

This set of solutions is not unique because the Wronskian is equal to zero.

Homogeneous Linear Equations

Homogeneous equations are differential equations with a forcing function equal to zero. Homogeneous equations are most often used as the first step when solving higher order equations. The solution to a homogeneous equation is called the complimentary solution. The form of the complimentary solution depends on the result of the auxiliary equation. An auxiliary equation is found by replacing y and its derivatives with an m to the power equal to the order of y. For example, if there is a y'', replace it with m^2, and if there is a y, replace it with $m^0 = 1$. The process for solving a homogeneous linear differential equation is given below:

Step 1: Find the auxiliary equation in the form $C_n m^n + C_{n-1} m^{n-1} + \cdots + C_1 m + C_0 = 0$.

Step 2: Solve for m by factoring.

Step 3: Use m to find the homogeneous equation's set of solutions $y_1 = C_1 e^{m_1 t}$ and $y_2 = C_2 e^{m_2 t}$ where C_1 and C_2 are unknown constants. Note that if m is imaginary, Euler's identity is used to find the form of the complimentary solution.

Step 4: Add the solutions together to find the general solution to the differential equation: $y = C_1 e^{m_1 t} + C_2 e^{m_2 t}$.

Step 5: If given, use the initial conditions to solve for the value of the unknown constants.

Example: Solve the following homogenous equation: $y'' + 6y' + 8y = 0$ given $y(0) = 2$ and $y'(0) = 1$

Step 1: Find the auxiliary equation in the form $C_n m^n + C_{n-1} m^{n-1} + \cdots + C_1 m + C_0 = 0$.

$$m^2 + 6m + 8 = 0$$

Step 2: Solve for m by factoring:

$$(m+4)(m+2)=0$$

$$m+4=0 \qquad\qquad m+2=0$$

$$m_1=-4 \qquad\qquad m_2=-2$$

Step 3: Use m to find the homogeneous equation's set of solutions:

$y_1 = C_1e^{m_1t}$ and $y_2 = C_2e^{m_2t}$ where C_1 and C_2 are unknown constants:

$$y_1 = C_1e^{-4t} \qquad\qquad y_2 = C_2e^{-2t}$$

Step 4: Add the solutions together to find the general solution to the differential equation $y = C_1e^{m_1t} + C_2e^{m_2t}$:

$$y = C_1e^{-4t} + C_2e^{-2t}$$

Step 5: If given, use the initial conditions to solve for the value of the unknown constants:

$$y = C_1e^{-4t} + C_2e^{-2t}, \quad y(0)=2 \qquad\qquad y' = -4C_1e^{-4t} - 2C_2e^{-2t}, \quad y'(0)=1$$

$$2 = C_1e^{-4(0)} + C_2e^{-2(0)} \qquad\qquad 1 = -4C_1e^{-4(0)} - 2C_2e^{-2(0)}$$

$$2 = C_1 + C_2 \qquad\qquad 1 = -4C_1 - 2C_2$$

$$C_1 = 2 - C_2$$

$$1 = -4(2-C_2) - 2C_2$$

$$1 = -8 + 2C_2$$

$$9 = 2C_2$$

$$C_2 = \frac{9}{2} = 4.5$$

$$C_1 = 2 - 4.5 = -2.5$$

The solution to the given homogeneous differential equation is $y = -2.5e^{-4t} + 4.5e^{-2t}$

Reduction of Order

When only one solution of a second order differential equation is known, reduction of order is used to find a second, unique solution. The process for using reduction of order is given below:

Step 1: Given a solution (y_1) to a second order homogeneous differential equation, declare a new solution $y_2 = \mu y_1$.

Step 2: Find the first and second derivatives of the new solution using the product rule.

Step 3: Substitute the derivatives into the ordinary differential equation and simplify the equation. Note that the μ terms will always cancel out.

Step 4: Declare a new function of x, $w = \mu' \rightarrow w' = \mu''$ to rewrite the equation as a first order differential equation.

Step 5: Solve the first order differential equation to find w.

Step 6: Integrate w to find μ and substitute it into the second solution. Note that y_2 will also include y_1, so the general solution is $y = y_2$.

Step 7: If given, use the initial conditions to solve for the value of the unknown constants.

Example: Find the second solution of the differential equation $y'' + 6y' + 9y = 0$ given that one solution is $y_1 = e^{-3x}$.

Step 1: Given a solution (y_1) to a homogeneous second order differential equation, declare a new solution $y_2 = \mu y_1$:

$$y_2 = \mu e^{-3x}$$

Step 2: Find the first and second derivatives of the new solution using the product rule:

$$y_2' = \mu' e^{-3x} - 3\mu e^{-3x}$$
$$y_2'' = \mu'' e^{-3x} - 3\mu' e^{-3x} - 3\mu' e^{-3x} + 9\mu e^{-3x} = \mu'' e^{-3x} - 6\mu' e^{-3x} + 9\mu e^{-3x}$$

Step 3: Substitute the derivatives into the ordinary differential equation and simplify the equation.

$$y'' + 6y' + 9y = 0$$
$$(\mu'' e^{-3x} - 6\mu' e^{-3x} + 9\mu e^{-3x}) + 6(\mu' e^{-3x} - 3\mu e^{-3x}) + 9(\mu e^{-3x}) = 0$$
$$(\mu'' - 6\mu' + 9\mu + 6\mu' - 18\mu + 9\mu)e^{-3x} = 0$$
$$(\mu'')e^{-3x} = 0$$

Step 4: Declare a new function of x, $w = \mu' \rightarrow w' = \mu''$ to rewrite the equation as a first order differential equation:

$$(w')e^{-3x} = 0$$

Step 5: Solve the first order differential equation to find w:

Since there is no value for x that satisfies the equation $e^{-3x} = 0$, w' must be equal to zero.

$$w' = 0$$

$$\int dw = \int 0dx$$

$$w = C_1$$

Step 6: Integrate w to find μ and substitute it into the second solution. Note that y_2 includes y_1, so the general solution is $y = y_2$.

$$\mu = \int wdw = \int C_1dx$$

$$\mu = C_1x + C_2$$

$$y_2 = \mu y_1 = y = (C_1x + C_2)e^{-3x} = C_1xe^{-3x} + C_2e^{-3x}$$

Step 7: If given, use the initial conditions to solve for the value of the unknown constants.

No initial conditions were given so the constants cannot be found.

Method of Undetermined Coefficients

Method of undetermined coefficients is used when a given linear differential equation contains a forcing function that is in an easily identifiable form such as a constant, polynomial, cosine, sine, or exponential. The form of the particular solution for different types of forcing functions are shown below:

Form of the forcing function: $f(x)$	Form of particular solution: y_p
C_1	A
$C_1x + C_2$	$Ax + B$
$C_1\sin(ax)$	$A\cos(ax) + B\sin(ax)$
$C_1\cos(ax)$	$A\cos(ax) + B\sin(ax)$
C_1e^{ax}	Ae^{ax}

Note that any of the constants in this table can be equal to zero. If the forcing function is a combination of different forms, the particular solution of each form will be added together. The process for using the method of undetermined coefficients is given below:

Step 1: Find the complementary solution by solving the equation's associated homogeneous equation.

Step 2: Based on the form of the forcing function, find the form of the equation's particular solution. Note, this can be found using the table above or in the textbook. The particular solution cannot match the complementary solution. If this occurs, multiply the particular solution by x.

Step 3: Find as many derivatives of the particular solution as the order of the differential equation.

Step 4: Substitute the derivatives into the differential equation and solve for the value of the constants.

Step 5: Create the general solution by adding the complementary and particular solutions together: $y = y_c + y_p$.

Step 6: If given, use the initial conditions to solve for the value of the unknown constants.

Example: Solve $y'' + 2y' - 15y = 30x - 19$

Step 1: Find the complementary solution by solving the associated homogeneous equation:

$$y'' + 2y' - 15y = 0$$

$$m^2 + 2m - 15 = 0$$

$(m + 5)(m - 3) = 0$; therefore, $m_1 = -5$ and $m_2 = 3$.

The complementary solution is $y_c = C_1e^{-5x} + C_2e^{3x}$.

Step 2: Based on the form of the forcing function, find the form of the particular solution.

Because the forcing function is a first order polynomial, the form of the particular solution is $Ax + B$.

Step 3: Find as many derivatives of the particular solution as the order of the differential equation:

$$y = Ax + B \qquad y' = A \qquad y'' = 0$$

Step 4: Substitute the derivatives into the differential equation, and solve for the value of the constants:

$$(0) + 2(A) - 15(Ax + B) = 30x - 19$$

$$-15Ax + (2A - 15B) = 30x - 19$$

$-15A = 30$ and $2A - 15B = -19$

$A = -2$ and $B = 1$

The particular solution is $y_p = -2x + 1$

Step 5: Create the general solution by adding the complementary and particular solutions together $y = y_c + y_p$:

$$y = C_1e^{-5x} + C_2e^{3x} - 2x + 1$$

Step 6: If given, use the initial conditions to solve for the value of the unknown constants.

No initial conditions were given, so the constants cannot be found.

Variation of Parameters

Variation of parameters can be used to solve a linear differential equation with any forcing function. This method is often more complicated than other methods, so using the method of undetermined coefficients is generally more efficient for differential equations with forcing functions that are in an easily identifiable form as defined above. Variation of parameters is accomplished by using Wronskians that include the forcing function to solve for a differential equation's particular solution. The process for using variation of parameters is given below:

Step 1: Find the complementary solution by solving the associated homogeneous equation.

Step 2: Solve for the Wronskians W, W_1, and W_2 where $f(x)$ is the forcing function and y_1 and y_2 are elements of the equation's complementary solution.

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

Step 3: Solve for u_1 and u_2.

$$u_1' = \frac{W_1}{W} \qquad u_2' = \frac{W_2}{W}$$

$$u_1 = \int u_1' \, du \qquad u_2 = \int u_2' \, du$$

Step 4: Create the particular solution $y_p = u_1 y_1 + u_2 y_2$.

Step 5: Create the general solution $y = y_c + y_p$.

Step 6: If given, use the initial conditions to solve for the value of the unknown constants.

Example: Solve the following differential equation: $y'' - 2y' + y = \frac{e^t}{t^2+1}$

Step 1: Find the complementary solution by solving the associated homogeneous equation:

$$m^2 - 2m + 1 = 0$$
$$(m-1)^2 = 0$$
$$y_1 = e^t \qquad y_2 = te^t$$
$$y_c = C_1 e^t + C_2 te^t$$

Step 2: Solve for the Wronskians W, W_1, and W_2, where $f(x)$ is the forcing function:

$$y_1' = e^t \qquad y_2' = te^t + e^t$$

$$W = \begin{vmatrix} e^t & te^t \\ e^t & te^t + e^t \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & te^t \\ \dfrac{e^t}{t^2+1} & te^t + e^t \end{vmatrix} \qquad W_2 = \begin{vmatrix} e^t & 0 \\ e^t & \dfrac{e^t}{t^2+1} \end{vmatrix}$$

$$W = e^{2t} \qquad W_1 = -\frac{te^{2t}}{t^2+1} \qquad W_2 = \frac{e^{2t}}{t^2+1}$$

Step 3: Solve for u_1 and u_2:

$$u_1' = \frac{-\dfrac{te^{2t}}{t^2+1}}{e^{2t}} = -\frac{t}{t^2+1}$$
$$u_1 = \int -\frac{t}{t^2+1}\,dt$$
$$u_1 = -\frac{1}{2}\ln\left(t^2+1\right)$$

$$u_2' = \frac{\dfrac{e^{2t}}{t^2+1}}{e^{2t}} = \frac{1}{t^2+1}$$
$$u_2 = \int \frac{1}{t^2+1}\,dt$$
$$u_2 = \tan^{-1}(t)$$

Step 4: Create the particular solution $y_p = u_1 y_1 + u_2 y_2$:

$$y_p = -\frac{1}{2}\ln(t^2+1)\,e^t + \tan^{-1}(t)\,te^t$$

Step 5: Create the general solution $y = y_c + y_p$:

$$y = C_1 e^t + C_2 te^t - \frac{1}{2}\ln(t^2+1)\,e^t + \tan^{-1}(t)\,te^t$$

<u>Step 6</u>: If given, use the initial conditions to solve for the value of the unknown constants.

No initial conditions were given so the constants cannot be found.