

## Logic and Truth Tables

### What is a Truth Table?

A truth table is a tool that helps you analyze statements or arguments in order to verify whether or not they are logical, or true. There are five basic operations that you will utilize when creating a truth table. These operations are the conjunction, disjunction, negation, conditional, and bi-conditional. These operations are also referred to as “and,” “or,” “not,” “if-then,” and “if and only if.” The rules for these operations are as follows:

### Logic Operations

AND $\wedge$ (conjunction)		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

**And Statements** – These statements are true **only** when **both** p and q are true (as the rigorous definition of “and” implies.)

*Example:* “I will bring both a pen AND a pencil to the tutoring session.” Only if I bring both is this true.

OR $\vee$ (disjunction)		
p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

**Or Statements** – These statements are false **only** when **both** p and q are false (follows the definition of “or”.)

*Example:* “I will bring a pen OR a pencil to the tutoring appointment.” Only if I don’t do either is this false.

NOT $\sim$ (negation)	
p	$\sim p$
T	F
F	T

**Not Statements** – The “not” is simply the **opposite** or **complement** of its original value.

*Example:* p = “It is raining”  
 $\sim p$  = “It is NOT raining”

IF-THEN (conditional)		
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

**If  $\rightarrow$  Then Statements** – These statements are false **only** when p is true and q is false (because anything can follow from a false premise.)

*Example:* “IF I am elected THEN taxes will go down.”

T	T	T
T	F	F
F	T	T
F	F	T

Only if I am elected and taxes don’t go down is this false.

IF AND ONLY IF (bi-conditional)		
p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

**If and Only If Statements** – These statements are true **only** when both p and q have the **same truth values**.

*Example:* “Taxes will go down IF AND ONLY IF I am elected.”

Only if I am elected and taxes go down, or I am not elected and taxes do not go down is this true.

### Constructing Truth Tables

To create a truth table, follow these steps:

- Determine the number of variables; for  $n$  variables, create a table with  $2^n$  rows.
  - If there are **two** variables (p, q), then you will need  $2^2$  or **4 rows**.
  - If there are **three** variables (p, q, and r), you will need  $2^3$  or **8 rows**.
- List the variable and every combination of T and F for the given variables.

TWO VARIABLE	
p	q
T	T
T	F
F	T
F	F

THREE VARIABLE		
p	q	r
T	T	T
T	T	F
T	F	T
T	F	F
F	T	T
F	T	F
F	F	T
F	F	F

- Then start with negations (“nots” or “~”), create a new column for each “piece” of the statement or argument, filling in the truth values as you go. Work from simple pieces to more difficult pieces until you have the truth values for the whole problem.
- If two statements have the same truth values, then they are equivalent.

### Equivalents

There are a number of equivalents in logic. This means that these statements have been proven true, and you can use these statements without having to prove them. The symbol for equivalent is  $\equiv$ . The following are the most commonly used equivalents:

- |  |   |   |
|--|---|---|
| • $p \wedge \mathbf{T} \equiv p$                     | • $p \wedge p \equiv p$                                     | • $\sim(p \vee q) \equiv \sim p \wedge \sim q$<br>DeMorgan’s Law #1 |
| • $p \wedge \mathbf{F} \equiv \mathbf{F}$            | • $p \vee p \equiv p$                                       | • $\sim(p \wedge q) \equiv \sim p \vee \sim q$<br>DeMorgan’s Law #2 |
| • $p \vee \mathbf{T} \equiv \mathbf{T}$              | • $p \wedge \sim p \equiv \mathbf{F}$                       |   |
| • $p \vee \mathbf{F} \equiv p$                       | • $p \vee \sim p \equiv \mathbf{T}$                         |   |
| • $p \rightarrow q \equiv \sim p \vee q$             | • $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ | • $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$       |
| • $p \rightarrow q \equiv \sim q \rightarrow \sim p$ |   |   |

### Truth Table Example

Statement:  $(p \wedge q) \leftrightarrow (\sim p \vee q)$

p	q
T	T
T	F
F	T
F	F

**Step 1:**  
Determine the number of variables and rows needed, then write down all possible combinations of  $p$  and  $q$ .

p	q	$\sim p$
T	T	F
T	F	F
F	T	T
F	F	T

**Step 2:**  
 $p \wedge q \leftrightarrow \sim p \vee q$   
Write down values of  $\sim p$  (or “not  $p$ ”)

p	q	$\sim p$	$p \wedge q$
T	T	F	T
T	F	F	F
F	T	T	F
F	F	T	F

**Step 3:**  
 $p \wedge q \leftrightarrow \sim p \vee q$   
Left Side: Is  $(p \wedge q)$  true? Use values from  $p$  and  $q$  columns.

p	q	$\sim p$	$p \wedge q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	F	T
F	F	T	F	T

**Step 4:**  
 $p \wedge q \leftrightarrow \sim p \vee q$   
Right Side: Is  $(\sim p \vee q)$  true? Use values from  $\sim p$  and  $q$  columns.

p	q	$\sim p$	$p \wedge q$	$\sim p \vee q$	$p \wedge q \leftrightarrow \sim p \vee q$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	F	T	F
F	F	T	F	T	F

**Step 5:**  
 $p \wedge q \leftrightarrow \sim p \vee q$   
Is the entire statement true? No.

The entire statement is true only when the last column’s truth values are all “True.” In this case,  $(p \wedge q)$  is **not** equivalent to  $(\sim p \vee q)$  because they do not have the same truth values.

If every cell in the last column is “True,” then the entire statement is true. This is called a **tautology**.

## Truth Tables for Arguments

A logical argument is made up of two parts: the **premises** and the **conclusion**. Arguments are usually written in the following form:

If it is cold, then my motorcycle will not start.  
My motorcycle started.  
 It is not cold.

Arguments of this form can be turned into a logical statement. First, assign variables to each premise and conclusion.

If it is cold, then my motorcycle will not start. <u>My motorcycle started.</u> It is not cold.	“It is cold” = $p$ “It is not cold” = $\sim p$ “My motorcycle will start” = $q$ “My motorcycle will not start” = $\sim q$
---	--

You can now re-write the argument using logical operators. Look for keywords like “if” and “not.”

<b>If</b> it is cold, then my motorcycle will <b>not</b> start. <u>My motorcycle started.</u> It is not cold.	$p \rightarrow \sim q$ <u><math>q</math></u> $\sim p$
---	---

Finally, the argument can be rewritten as a logical statement. Arguments can always be written in the following form:

$$[\text{Premise One} \wedge \text{Premise Two}] \rightarrow \text{Conclusion}$$

Our example would be rewritten as:

$$[(p \rightarrow \sim q) \wedge q] \rightarrow \sim p$$

This statement can then be proven valid or invalid using a truth table, as shown on the following page.

### Truth Table Example

Statement:  $[(p \rightarrow \sim q) \wedge q] \rightarrow \sim p$

p	q
T	T
T	F
F	T
F	F

**Step 1:**

Determine the number of variables and rows needed, then write down all possible combinations of  $p$  and  $q$ .

p	q	$\sim p$
T	T	F
T	F	F
F	T	T
F	F	T

**Step 2:**

$[(p \rightarrow \sim q) \wedge q] \rightarrow \sim p$   
Write down values of  $\sim p$  (or “not  $p$ ”)

p	q	$\sim p$	$\sim q$
T	T	F	F
T	F	F	T
F	T	T	F
F	F	T	T

**Step 3:**

$[(p \rightarrow \sim q) \wedge q] \rightarrow \sim p$   
Write down values of  $\sim q$  (or “not  $q$ ”)

p	q	$\sim p$	$\sim q$	$(p \rightarrow \sim q)$
T	T	F	F	F
T	F	F	T	T
F	T	T	F	T
F	F	T	T	T

**Step 4:**

$[(p \rightarrow \sim q) \wedge q] \rightarrow \sim p$   
Left Side: Start within the inner most parentheses. Is  $(p \rightarrow \sim q)$  true? Use values from  $p$  and  $\sim q$  columns.

p	q	$\sim p$	$\sim q$	$(p \rightarrow \sim q)$	$(p \rightarrow \sim q) \wedge q$
T	T	F	F	F	F
T	F	F	T	T	F
F	T	T	F	T	T
F	F	T	T	T	F

**Step 5:**

$[(p \rightarrow \sim q) \wedge q] \rightarrow \sim p$   
Left Side: Work outwards towards the outer most parentheses. Is  $[(p \rightarrow \sim q) \wedge q]$  true? Use values from  $(p \rightarrow \sim q)$  and  $q$  columns.

p	q	$\sim p$	$\sim q$	$(p \rightarrow \sim q)$	$(p \rightarrow \sim q) \wedge q$	$[(p \rightarrow \sim q) \wedge q] \rightarrow \sim p$
T	T	F	F	F	F	T
T	F	F	T	T	F	T
F	T	T	F	T	T	T
F	F	T	T	T	F	T

**Step 6:**

$[(p \rightarrow \sim q) \wedge q] \rightarrow \sim p$   
Is the entire statement true?  
Yes. This is a tautology