Factoring Methods

Factoring is a process used to solve algebraic expressions. An essential aspect of factoring is learning how to find the greatest common factor (GCF) of a given algebraic problem. Once the GCF is determined, students will be able to simplify a given expression into a solvable form. This handout will explain how to find the greatest common factor as well as demonstrate the following methods of factoring: grouping, slide and divide, difference of perfect squares, sum and difference of cubes, and substitution.

You can navigate to specific sections of this handout by clicking the links below.

Finding the Greatest Common Factor (GCF)

A greatest common factor (GCF) is the largest number or variable that can be evenly divided from each term within an algebraic expression. When solving algebraic expressions, always check for a common factor. If there is one, factor out the GCF before trying to factor with any other method. This can be done by breaking an expression's terms into the smallest factors possible. Next, the GCF should be moved from the original terms so that it is multiplied by the remaining expression. The number 1 cannot serve as a GCF.

GCF Steps

1. Break the terms into the smallest factors possible to determine the GCF.
2. Factor out the GCF.

Example 1: Determine the GCF and factor.

$$5x^4 - 35x^3 + 10x^2$$

Step 1: Break the terms into the smallest factors possible to determine the GCF.

$5x^4 = (\mathbf{5})(\mathbf{x})(\mathbf{x})(x)(x)$

$35x^3 = (7)(\mathbf{5})(\mathbf{x})(\mathbf{x})(x)$

$10x^2 = (\mathbf{5})(2)(\mathbf{x})(\mathbf{x})$

The GCF is $(\mathbf{5})(\mathbf{x})(\mathbf{x})$ or $\mathbf{5x^2}$.

Step 2: Factor out the GCF.

$\boxed{\mathbf{5x^2}(x^2 - 7x + 2)}$

Factoring by Grouping

If a four-term polynomial is present, and there is no GCF shared by all four terms, the terms can be grouped into pairs that have a GCF. This method is called factoring by grouping.

Grouping Steps

1. Check for a GCF.
2. Group the terms so that two identical sets of parentheses are left after factoring.
3. Factor out the new GCF.
4. Replace the brackets with parentheses.

Example 2: Factor the following expression using the grouping method.

$$10abx - 8ax + 15bx - 12x$$

Step 1: Check for a GCF. Every term in the expression has an x, so it is the first GCF.

$\mathbf{x}(10ab - 8a + 15b - 12)$

Step 2: Group the terms so that two identical sets of parentheses are left after factoring. The matching parentheses will become the new GCF. Rearrange the terms if the parentheses sets do not match.

$x[(10ab + 15b) + (-8a - 12)]$

$x[5b(\mathbf{2a + 3}) + -4(\mathbf{2a + 3})]$

Step 3: Factor out the new GCF. The GCF is the matching set of parentheses.

$x(\mathbf{2a + 3})[5b - 4]$

Step 4: Replace the brackets with parentheses.

$\boxed{x(2a + 3)(5b - 4)}$

Additionally, at least one group must share a common factor other than the number one. These conditions are not met in Example 3A, so the groups must be changed. If the terms cannot be rearranged to find a GCF, the expression is not factorable by grouping.

Example 3:

Factor the following expression using the grouping method.

3A. $6xy - 5 - 15x + 2y$

$(6xy - 5) + (-15x + 2y)$

$1(6xy - 5) + 1(-15x + 2y)$

3B. $6xy - 5 - 15x + 2y$

$(6xy - 15x) + (-5 + 2y)$

$3x(2y - 5) + 1(-5 + 2y)$

$\boxed{(2y - 5)(3x + 1)}$

Factoring by Grouping: Quadratic Expressions

Factoring by grouping can also be used to factor problems in the form $ax^2 + bx + c$. The letters a, b, and c represent numbers, and their order in the expression can vary (i.e. $bx + ax^2 + c$). If there is no number in front of an x term, then the number is 1. When *a* is not 1, another factoring method mentioned later in this handout may need to be used.

Grouping Steps: Quadratics

1. Identify the values that will represent *a*, *b*, and *c*.
2. Find the factors of *c*.
3. Determine which factors of c add up to equal *b*.
4. Replace the *b* term of the original expression with the chosen factors.
5. Group the new expression into pairs.
6. Factor out common terms.
7. Simplify.

Example 4: Factor the following polynomial using the grouping method.

$$x^2 - 5x + 6$$

Step 1: Identify the values that will represent *a*, *b*, and *c*.

$a = 1 \quad b = -5 \quad c = 6$

Step 2: Find the factors of *c*.

Factors		Multiply to c
1	6	$1 \times 6 = 6$
-1	-6	$(-1) \times (-6) = 6$
2	3	$2 \times 3 = 6$
-2	-3	$(-2) \times (-3) = 6$

Step 3: Determine which factors add up to equal *b*.

Factors		Sum to b
-2	-3	$(-2) + (-3) = -5$

Step 4: Replace the *b* term of the original expression with the chosen factors.

$x^2 - \mathbf{3x} - \mathbf{2x} + 6$

Step 5: Group the new expression into pairs.

$(x^2 - 3x) + (-2x + 6)$

Step 6: Factor out common terms.

$x(x - 3) + -2(x - 3)$

Step 7: Simplify.

$\boxed{(x - 3)(x - 2)}$

Factoring by the Slide and Divide Method

The slide and divide method is used when the *a* term is not 1 in a three-term polynomial. This method converts the *a* term to 1, making it easier to factor.

Slide and Divide Steps

1. Identify the values of *a*, *b*, and *c*.
2. Slide *a* over to be multiplied by *c*.
3. Identify the modified values of *a*, *b*, and *c*.
4. Find the factors of *c*.
5. Determine which factors add up to equal *b*.
6. Replace the *b* term of the original expression with the chosen factors.
7. Group the new expression into pairs.
8. Factor out common terms.
9. Rewrite the factors.
10. Divide both new factors by the *a* value of the original expression.
11. Simplify.

Example 5: Factor the following quadratic using the slide and divide method.

$$2x^2 + 5x + 3$$

Step 1: Identify the values of *a*, *b*, and *c*.

$a = 2 \quad b = 5 \quad c = 3$

Step 2: Slide *a* over to be multiplied by *c*. This reduces *a* to 1 and allows one of the other factoring methods to be used.

$2x^2 + 5x + 3$

$(2) \times (3) = 6$

Expression: $x^2 + 5x + 6$

Step 3: Identify the modified values of *a*, *b*, and *c*.

$a = 1 \quad b = 5 \quad c = 6$

<u>Step 4:</u> Find the factors of *c*.

Factors		Multiply to c
1	6	$1 \times 6 = 6$
-1	-6	$(-1) \times (-6) = 6$
2	3	$2 \times 3 = 6$
-2	-3	$(-2) \times (-3) = 6$

<u>Step 5:</u> Determine which factors add up to equal *b*.

Factors		Sum to b
2	3	$2 + 3 = 5$

<u>Step 6:</u> Replace the *b* term of the original expression with the chosen factors.

$$x^2 + 2x + 3x + 6$$

<u>Step 7:</u> Group the new expression into pairs.

$$(x^2 + 2x) + (3x + 6)$$

<u>Step 8:</u> Factor out common terms.

$$x(x + 2) + 3(x + 2)$$

<u>Step 9:</u> Rewrite the factors.

$$(x + 2)(x + 3)$$

<u>Step 10:</u> Divide both new factors by the *a* value in the original expression.

$$a = 2 \qquad \left(x + \frac{2}{2}\right)\left(x + \frac{3}{2}\right)$$

<u>Step 11:</u> Simplify. Because 3 is not evenly divided by 2, move the 2 over to be multiplied by *x*. Thus, if *a* does not evenly divide into a term's numerator, move it over to be multiplied by the first term instead.

$$\boxed{(x + 1)(2x + 3)}$$

Factoring by the Difference of Perfect Squares

When working with expressions containing perfect squares, the difference of perfect squares formula can be used. Perfect squares are numbers times themselves. An example is 9, which is 3×3, or 3^2.

The Difference of Perfect Squares Formula

Difference of Perfect Squares
$a^2 - b^2 = (a + b)(a - b)$

Example 6: Factor using the difference of perfect squares formula: $a^2 - b^2 = (a + b)(a - b)$.

$$x^2 - 4$$

Step 1: Factor both terms in the expression to see if they are perfect squares.

$x^2 = x \times x$ and $4 = 2 \times 2 = 2^2$ or b^2

Step 2: Both x^2 and 4 are perfect squares, so the second term can be replaced by b^2.

$x^2 - 2^2$

Step 3: Break the terms up according to the formula.

$\boxed{(x + 2)(x - 2)}$

Factoring Cubic Expressions

When working with expressions containing perfect cubes, the sum of cubes and difference of cubes formulas can be used. If the cubed terms are being added, use the sum of cubes formula. If the cubed terms are being subtracted, use the difference of cubes formula.

The Sum and Difference of Cubes Formulas

Sum of Cubes	Difference of Cubes
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Example 7: Factor using the correct cube formula.

$$x^3 + 27$$

Step 1: Break the terms apart to see if they are perfect cubes.

$x^3 = x \times x \times x$ and $27 = 3 \times 3 \times 3 = 3^3$ so… $x^3 + 3^3$

Step 2: Both terms are perfect cubes, and there is a plus sign between the two original terms, so use the sum of cubes formula: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.

$\boxed{(x + 3)(x^2 - 3x + 9)}$

The SOFAS Method

SOFAS stands for square, opposite, first, always positive, and square the back. It is another method for factoring when cubed terms are present in an expression.

SOFAS Steps

1. Cube root both terms to obtain the first factor.
2. For the second factor, **S**quare the first term of the first factor.
3. Place the **O**pposite sign of what is between the terms in the first factor.
4. Multiply the **F**irst term by the last term.
5. **A**lways positive: add a positive sign to the expression.
6. **S**quare the back term.
7. Ensure that all parts of the expression are put together.

Example 8: Factor this expression using the SOFAS method.

$$x^9 + 216$$

Step 1: Cube root both terms to obtain the first factor.

Note: The second factor is dependent on the terms in the first factor.

$\sqrt[3]{x^9} = x^3$ and $\sqrt[3]{216} = 6$

Expression: $(x^3 + 6)(\qquad\qquad)$

Step 2: For the second factor, **S**quare the first term of the first factor.

$(x^3)^2 = x^6$

Expression: $(x^3 + 6)(x^6 \qquad)$

Step 3: Place the **O**pposite sign of what is between the terms in the first factor.

$x^6 -$

Expression: $(x^3 + 6)(x^6 - \qquad)$

Step 4: Multiply the **F**irst term by the last term.

$x^3 \times 6 = 6x^3$

Expression: $(x^3 + 6)(x^6 - 6x^3 \qquad)$

Step 5: **A**lways positive: add a positive sign to the expression.

$x^6 - 6x^3 +$

Expression: $(x^3 + 6)(x^6 - 6x^3 + \quad)$

Step 6: **S**quare the back term.

$6^2 = 36$

Expression: $(x^3 + 6)(x^6 - 6x^3 + 36)$

Step 7: Ensure that all parts of the expression are put together.

$\boxed{(x^3 + 6)(x^6 - 6x^3 + 36)}$

Factoring by Substitution

Substitution is used to factor expressions with large exponents. The goal is to simplify the expression so that one of the other factoring methods can be used.

Substitution Steps

1. Identify the largest variable that is squared or cubed.
2. Substitute that term with a different variable.
3. Factor using one of the other factoring methods.
4. Reverse the substitution.
5. Simplify.

Example 9: Factor the following expression using substitution.

$$x^4 - 4x^2 - 45$$

Step 1: Identify the largest variable that is squared or cubed.

$x^4 = (x^2)^2$

Step 2: Substitute that term with a different variable, in this case *u*.

$u = x^2$

Expression: $u^2 - 4u - 45$

Step 3: Factor using one of the other factoring methods.

Note: The grouping method was used here. Please review page 2 for information on the grouping method.

$a = 1 \quad b = -4 \quad c = -45$

Factors **Multiply to c**

1	-45	$1 \times (-45) = -45$
-1	45	$(-1) \times 45 = -45$
3	-15	$3 \times (-15) = -45$
-3	15	$(-3) \times 15 = -45$
5	-19	$5 \times (-19) = -45$
-5	19	$(-5) \times 19 = -45$

Factors **Sum to b**

5	-9	$5 + (-9) = -4$

$u^2 - 9u + 5u - 45$

$(u^2 - 9u) + (5u - 45)$

$u(u - 9) + 5(u - 9)$

$(u + 5)(u - 9)$

Step 4: Reverse the substitution.

Substitute the x^2 back into the expression.

$(x^2 + 5)(x^2 - 9)$

Step 5: Simplify.

Factor $x^2 - 9$ using the difference of perfect squares formula discussed on page 6.

$\boxed{(x^2 + 5)(x + 3)(x - 3)}$

The quadratic formula and the complete the square method can also be used for factoring algebraic expressions. Please refer to the Academic Center for Excellence's The Quadratic Formula and the Discriminant and Complete the Square handouts on our website at https://germanna.edu/academic-center-excellence/helpful-handouts.

Sample Problems:

1. $2x^2 - 18$
2. $3y^2 - 48$
3. $a^4 - 16$
4. $5a^2 - 30a + 45$
5. $4a^2 + 16a + 16$
6. $-x^2 + 50x - 625$
7. $ax - bx + ay - by$
8. $2ax + 3 + x + 6a$
9. $m^3 + n^3$
10. $r^3 - s^3$
11. $64x^3 - 1$
12. $8x^3 + 1$
13. $27y^3 - 1$
14. $125y^3 - 1$
15. $3a^2 - 2ax - 3a + 2x$
16. $a^2 - 2a + ab - 2b$
17. $125 - y^3$
18. $x^6 - 27$
19. $x^6 + 125$
20. $m^3n^6 + d^3$
21. $a^3 - a^2b - a + b$
22. $x^2 + 6x + 5$
23. $x^2 - 4x + 3$
24. $n^2 + 5n + 6$
25. $n^2 - 10n + 25$
26. $m^2 + 3ms - 4s^2$
27. $y^2 + 4y - 12$
28. $y^2 - y - 30$
29. $t^2 - 14t - 72$
30. $6 - x - x^2$
31. $36 + 5x - x^2$
32. $36s^2 + 12s + 1$
33. $6x^2 + 30x - 900$
34. $2a^4 - 10a^3 - 72a^2$
35. $2x^3 - 3x^2 - 2x + 3$
36. $(x - 1)^2 - 4$
37. $(x + 2)^2 - (y - 3)^2$
38. $16 - (2x - 1)^2$
39. $4a^2 - 4ab - 36 + b^2$
40. $2a^3 - 16a^2 + 32a$

Answers:

1. $2(x-3)(x+3)$
2. $3(y-4)(y+4)$
3. $(a-2)(a+2)(a^2+4)$
4. $5(a-3)^2$
5. $4(a+2)^2$
6. $-(x-25)^2$
7. $(x+y)(a-b)$
8. $(x+3)(2a+1)$
9. $(m+n)(m^2-mn+n^2)$
10. $(r-s)(r^2+rs+s^2)$
11. $(4x-1)(16x^2+4x+1)$
12. $(2x+1)(4x^2-2x+1)$
13. $(3y-1)(9y^2+3y+1)$
14. $(5y-1)(25y^2+5y+1)$
15. $(a-1)(3a-2x)$
16. $(a-2)(a+b)$
17. $(5-y)(25+5y+y^2)$
18. $(x^2-3)(x^4+3x^2+9)$
19. $(x^2+5)(x^4-5x^2+25)$
20. $(mn^2+d)(m^2n^4-mn^2d+d^2)$
21. $(a-1)(a+1)(a-b)$
22. $(x+5)(x+1)$
23. $(x-3)(x-1)$
24. $(n+2)(n+3)$
25. $(n-5)^2$
26. $(m-s)(m+4s)$
27. $(y+6)(y-2)$
28. $(y-6)(y+5)$
29. $(t-18)(t+4)$
30. $-(x-2)(x+3)$
31. $-(x-9)(x+4)$
32. $(6s+1)^2$
33. $6(x+15)(x-10)$
34. $2a^2(a-9)(a+4)$
35. $(x-1)(x+1)(2x-3)$
36. $(x+1)(x-3)$
37. $(x-y+5)(x+y-1)$
38. $-(2x-5)(2x+3)$
39. $(2a-b+6)(2a-b-6)$
40. $2a(a-4)^2$