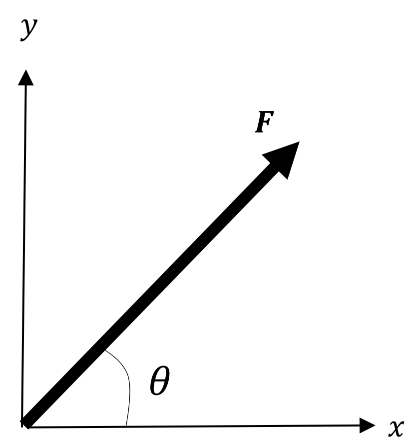
# Vectors in Two Dimensions

## Introduction

In engineering, physics, and mathematics, vectors are a mathematical or graphical representation of a physical quantity that has a magnitude as well as a direction. Both magnitude and direction are required to define a vector. A force vector, for example, will have both a magnitude (a scalar quantity such as 10 Newtons) and a direction (up, down, left, right, $30^o$ from the horizontal, etc.). While using vectors in three dimensional space is more applicable to the real world, it is far easier to learn vectors in two dimensional space first. This handout will only focus on vectors in *two dimensions*.

In two dimensional space, ($R^2$), a vector can be represented graphically as an arrow with a starting point and an ending point. The length of the arrow represents the magnitude of the vector, while the direction in which the arrow is pointing represents the vector's direction. The negative, or opposite of a vector is a vector with the same magnitude as the original, and the opposite direction. In this handout, vectors will be distinguished by using a bold font, for example, the force vector: $\boldsymbol{F}$**.** The magnitude of a vector will be written as the absolute value of that vector: $|\boldsymbol{F}|$. The figure below shows an arbitrary vector in $R^2$, where its magnitude is defined by its length $\boldsymbol{F}$, and its direction is defined by the angle $\theta$.



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**Adding Vectors Geometrically**

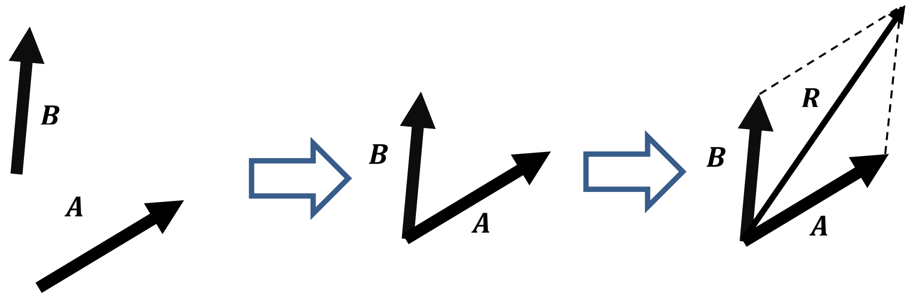
Vectors can be added using one of two methods: the parallelogram method or the tail-to-tip method. Vector addition is commutative; therefore, $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$.

To add vectors using the parallelogram method, two arbitrary vectors (**A** and **B**) can be moved so that their tails are coincident (their tales share the same point). A parallelogram is then drawn using the two vectors. The sum of the two vectors (**R**) is the vector drawn from the coincident tails to the opposite vertex of the parallelogram.

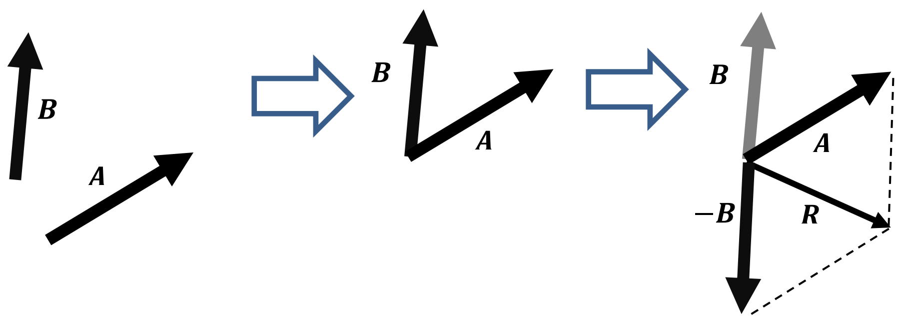
Since a vector is defined by its magnitude and direction, vectors can be translated to any position as long as the magnitude and direction of the vector remain the same.

Subtracting a vector from another can be seen as adding a negative vector.

**Example A:** $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{R}$

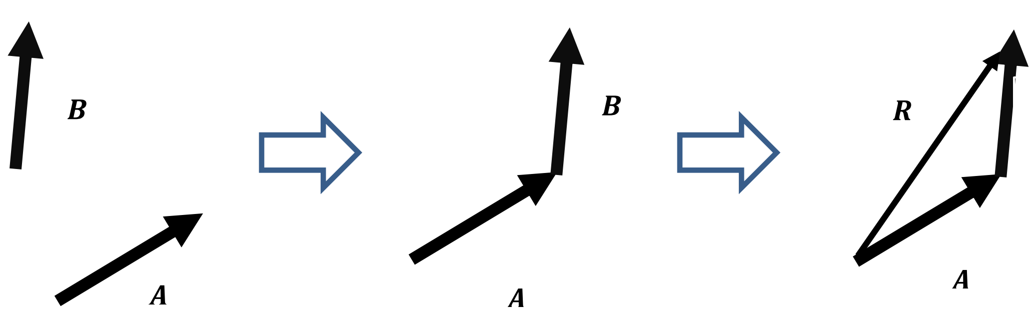



**Example B:** $\boldsymbol{A} - \boldsymbol{B}\ \big(\ or\ \boldsymbol{A} + (-\boldsymbol{B})\big) = \boldsymbol{R}$




To add two vectors using the tail-to-tip method, take the tail of one vector ($\boldsymbol{B}$), and move it so it is coincident with the tip of the other vector ($\boldsymbol{A}$). Then draw a vector with its tail coincident with the tail of the first vector ($\boldsymbol{A}$) and its tip coincident with the tip of the second vector ($\boldsymbol{B}$). This vector is the resultant vector ($\boldsymbol{R}$).

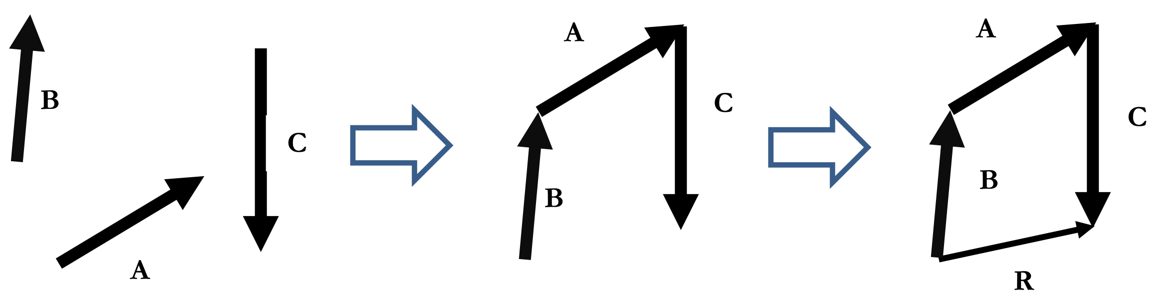
**Example C:** $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{R}$



When trying to add three or more vectors together, it is possible to use the parallelogram method. First add two vectors, then add the resultant to the third vector. However, this method can become somewhat hectic and time consuming; therefore, the tail-to-tip method is preferred.

To add three or more vectors using the tail-to-tip method, follow the steps in the previous problem while adding another vector.

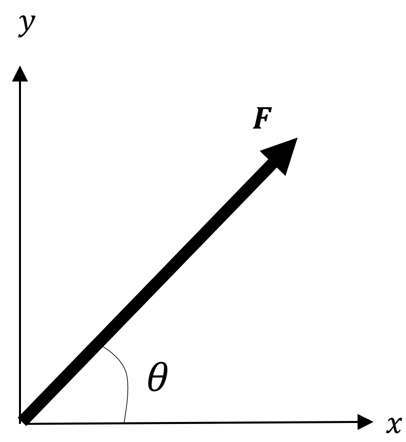
**Example D:** $\boldsymbol{B} + \boldsymbol{A} + \boldsymbol{C} = \boldsymbol{R}$




**Components of Vectors**

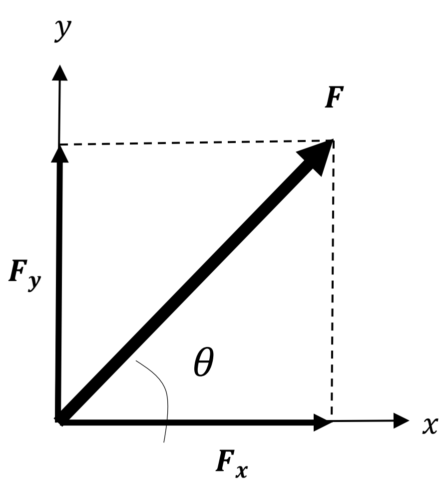
A useful way to describe the direction or orientation of a vector is by projecting it onto a set of axes. These axes can be arbitrary, but it is often easier to use the principal axes (the $x$, $y$ and $z$-axis). The process of projecting a vector onto the principal axes is often called *resolving the vector into its components*. This process can be thought of as finding the vector's shadow on the principal axis.

**Example E:** Find the components of the given vector, **F**




Step One

Draw a two dimensional vector defined by its magnitude $\boldsymbol{F}$, and its direction, $\theta$. The answer to this example will change depending on which quadrant the vector is drawn in. In this example the vector is drawn in the first quadrant and the direction is measured counterclockwise from the $x$-axis.

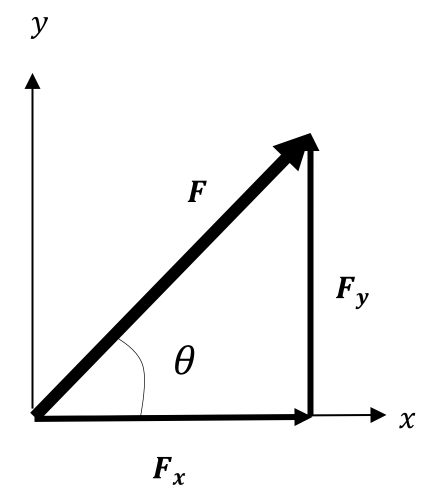



Step Two

To resolve this vector into its components projected onto the $x$- and $y$-axis, draw its "shadow" on the $x$- and $y$-axis ($\boldsymbol{F_x}$ and $\boldsymbol{F_y}$, respectively).

Using the parallelogram law of vector addition, it can be seen that $\boldsymbol{F_x}$ and $\boldsymbol{F_y}$ add to equal $\boldsymbol{F}$.

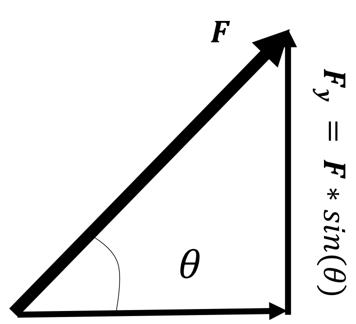
**Example E: (continued)**:




Step Three

Since vectors can be moved from one position to another as long as the magnitude and direction are not altered, move either component of $\boldsymbol{F}$ to form a right triangle with the original vector $\boldsymbol{F}$.

In this example the $y$-component was moved to form a right triangle with $\boldsymbol{F}$. Moving the $x$- component would also work, but it would require more calculations and produce the answer in a more complex form.




$\boldsymbol{F_x} = \boldsymbol{F} * cos(\theta)$

Step Four

Since the length of a vector is represented by its magnitude, right triangle trigonometry can be used to find the magnitude of $\boldsymbol{F_x}$ and $\boldsymbol{F_y}$ in terms of $\boldsymbol{F}$ and □, right triangle trigonometry is used:

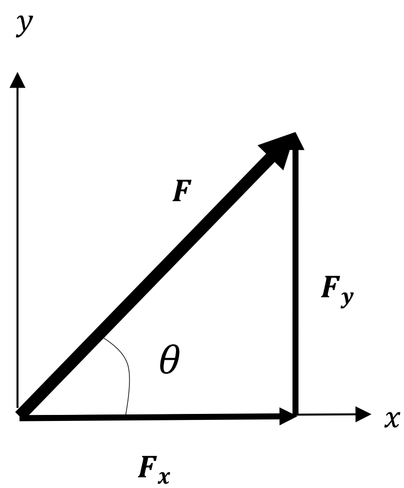
$$cos(\theta) = \frac{adjacent}{hypoteneuse} = \frac{\boldsymbol{F_x}}{\boldsymbol{F}}$$ Definition of cosine

$$\boldsymbol{F} * cos(\theta) = \frac{\boldsymbol{F_x}}{\boldsymbol{F}} * \boldsymbol{F}$$ Multiply both sides of the equation by **F** and then simplify

$$\boxed{\boldsymbol{F_x} = \boldsymbol{F} * cos(\theta)}$$ Solution

The same method is used to find that $\boldsymbol{F_y} = \boldsymbol{F} * sin(\theta)$ for this specific process and vector.

**Example F:** Write the magnitude of the vector $\boldsymbol{F}$ used in **Example E**, in terms of its components: $\boldsymbol{F_x}$ and $\boldsymbol{F_y}$.




Step One

Recall that the $x$ and $y$ components of a vector can be moved to produce a right triangle with the original vector.

Step Two

Because the magnitude of a vector is represented by its length, the Pythagorean Theorem can be used to calculate the magnitude of vector $\boldsymbol{F}$.

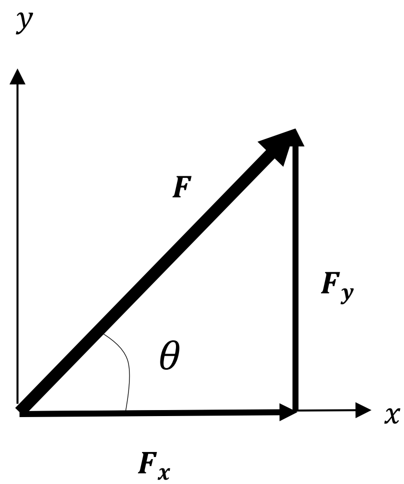
$$\boldsymbol{F}^2 = (\boldsymbol{F_x})^2 + (\boldsymbol{F_y})^2$$

Solution

Solve for $\boldsymbol{F}$, then simplify to find the solution. Since the magnitude of $\boldsymbol{F}$ is written as $|\boldsymbol{F}|$, the solution is written in the form "$|\boldsymbol{F}| =$…".

$$|\boldsymbol{F}| = \sqrt{(\boldsymbol{F_x})^2 + (\boldsymbol{F_y})^2}$$

**Example G:** Write the direction of the vector $\boldsymbol{F}$ used in **Example E** and **F**, in terms of its components $\boldsymbol{F_x}$ and $\boldsymbol{F_y}$.




<u>Step One</u>
The $x$ and $y$ components of $\boldsymbol{F}$ are moved to form a right triangle.

$$\tan(\theta) = \frac{opposite}{adjacent}$$

<u>Step Two</u>
Recall the definition of tangent in a right triangle.

$$\tan(\theta) = \frac{\boldsymbol{F_y}}{\boldsymbol{F_x}}$$

<u>Step Three</u>
Substitute the given values into the equation of tangent.

$$tan^{-1}(\tan(\theta)) = \tan^{-1}(\frac{\boldsymbol{F_y}}{\boldsymbol{F_x}})$$

<u>Step Four</u>
Take the inverse tangent of both sides of the equation, then simplify.

$$\boxed{\theta = \tan^{-1}(\frac{\boldsymbol{F_y}}{\boldsymbol{F_x}})}$$

<u>Solution</u>

**Unit Vectors**

A unit vector is a vector with a magnitude of one. Unit vectors are used to describe a direction, but not a specific magnitude. In this handout, unit vectors will be written the same way as regular vectors, but with a hat on top of it them. The hat signifies that the vector has a magnitude of one. An example of writing a unit vector with a hat on top of it would be the unit vector $\hat{\boldsymbol{u}}$.

Since describing the direction of the $x$- and $y$-axes is useful, there are special unit vectors that describe these directions. The unit vector that describes the $x$-direction is the $\hat{\boldsymbol{\imath}}$ vector, and the unit vector that describes the $y$-direction is the $\hat{\boldsymbol{\jmath}}$ vector.

To calculate a vector that points in the direction of an arbitrary vector but has a magnitude of one, divide each individual component by the magnitude of the vector.

The unit vector of an arbitrary vector $\boldsymbol{F}$ can be written mathematically as $\hat{\boldsymbol{u}} = \frac{\boldsymbol{F}}{|\boldsymbol{F}|}$

**Example H:** Find the unit vector that points in the same direction as the velocity vector $\boldsymbol{V} = 3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$

$$|\boldsymbol{V}| = \sqrt{3^2 + 4^2} = 5$$

Step One
Use the Pythagorean Theorem to find that the magnitude of $\boldsymbol{V}$ is 5.

$$\hat{\boldsymbol{u}} = \frac{3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}}{5}$$

Step Two
Divide the vector **V** by its magnitude, 5, then simplify.

$$\boxed{\hat{\boldsymbol{u}} = \frac{3}{5}\hat{\boldsymbol{\imath}} + \frac{4}{5}\hat{\boldsymbol{\jmath}}}$$

Solution

**Cartesian Vector Notation**

The unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are very useful and can be used to write a vector in terms of its magnitude and direction. For example, say there is a force vector, $\boldsymbol{F}$, described by its components $\boldsymbol{F_x} = 25N$ and $\boldsymbol{F_y} = 10N$.

Utilizing vector addition, and the fact that $\boldsymbol{F_x} + \boldsymbol{F_y} = \boldsymbol{F}$, the vector can be rewritten in a way that uses unit vectors. The first step in this process is to take each of the components, and multiply their given magnitude by the unit vector that describes their directions; for this example, multiply the magnitude of the $x$-component of $\boldsymbol{F}$ by the unit vector that describes the $x$-direction to get $25 * \hat{\boldsymbol{\imath}}$ (or $25\hat{\boldsymbol{\imath}}$) for the $x$-component. The same is done to rewrite the $y$-component as $10\hat{\boldsymbol{\jmath}}$.

$$\boldsymbol{F_x} = (25\hat{\boldsymbol{\imath}})N \qquad\qquad \boldsymbol{F_y} = (10\hat{\boldsymbol{\jmath}})N$$

Since the components of $\boldsymbol{F}$ add to equal $\boldsymbol{F}$, the vector can be written in terms of its components:

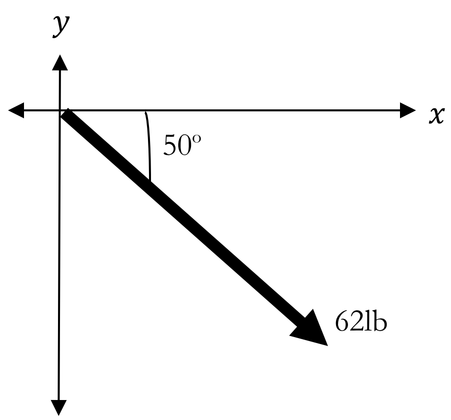
$$\boldsymbol{F} = (25\hat{\boldsymbol{\imath}} \ + \ 10\hat{\boldsymbol{\jmath}})N.$$

This form of writing a vector is called Cartesian Vector Notation.

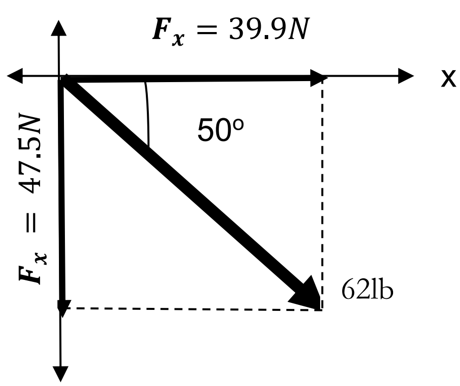
An alternate way to write a vector in terms of its components is a by using angle brackets to contain the values of the components. The alternate notation does not use Cartesian Vectors to denote which value corresponds to which component, instead it uses a strict format. The format for the alternate notation is: $\boldsymbol{F} =< \boldsymbol{F_x}, \boldsymbol{F_y}>$.

For example, the vector $\boldsymbol{F} = (25\hat{\boldsymbol{\imath}} \ + \ 10\hat{\boldsymbol{\jmath}})N$ can be written as $\boldsymbol{F} =< 25, 10 > N$ using the alternate notation.

**Example I:** Given a force, $\boldsymbol{F}$, with a magnitude of 62 pounds and a direction 50 degrees clockwise from the positive $x$-axis, write the components of $\boldsymbol{F}$ in Cartesian Vector Notation.




<u>Step One</u>
Begin by drawing the vector in the $xy$-plane




<u>Step Two</u>
Find the magnitude and direction of each component of $\boldsymbol{F}$:

$\boldsymbol{F_x}$: <u>Magnitude</u>: $62 * cos(50^o)N = 39.9N$
<u>Direction</u>: positive $x$-direction, or the $\hat{\boldsymbol{\imath}}$ direction

$\boldsymbol{F_y}$: <u>Magnitude</u>: $62 * sin(50^o)N = 47.5N$
<u>Direction</u>: negative $y$-direction, or the $-\hat{\boldsymbol{\jmath}}$ direction

$$\boldsymbol{F} = 39.9\hat{\boldsymbol{\imath}} - 47.5\hat{\boldsymbol{\jmath}}$$

<u>Solution</u>
By finding the magnitude and direction of each individual component, this vector can be written in Cartesian Vector Notation.

**Adding Vectors Mathematically**

While adding vectors using the parallelogram and tail-to-tip method is useful because it provides a general idea of what the result of adding two or more vectors will look like, vectors can also be added mathematically by utilizing Cartesian Vector Notation.

**Example J:** Given two vectors written in Cartesian Vector Notation: $\boldsymbol{F} = 2\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}$ and $\boldsymbol{P} = 5\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$, find the sum of $\boldsymbol{F}$ and $\boldsymbol{P}$.

$$\boldsymbol{F} + \boldsymbol{P} = (\boldsymbol{F_x} + \boldsymbol{F_y}) + (\boldsymbol{P_x} + \boldsymbol{P_y}) = \boldsymbol{R}$$

Step One

Rewrite the vectors in terms of their components. The sum of $\boldsymbol{F}$ and $\boldsymbol{P}$ will be the resultant vector $\boldsymbol{R}$.

$$\boldsymbol{F} + \boldsymbol{P} = (2\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}}) + (5\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}) = \boldsymbol{R}$$

Step Two

Substitute in the values for $\boldsymbol{R}$ and $\boldsymbol{F}$, written in Cartesian Vector Notation, then simplify by adding the components of each vector; that is the $x$-component of $\boldsymbol{F}$ with the $x$-component of $\boldsymbol{P}$, and the $y$-component of $\boldsymbol{F}$ with the $y$-component of $\boldsymbol{P}$.

$$\boldsymbol{R} = 7\hat{\boldsymbol{\imath}} + 7\hat{\boldsymbol{\jmath}}$$

Solution

**One very common mistake when adding vectors mathematically is to add the values of the $x$- and $y$- components together**. This is something that should not be done because it defeats the purpose of breaking the vector into its components before adding them. Adding the $x$- and $y$-components is much like adding $2x$ and $3y$, the result is $2x + 3y$, not $5x$ or $5y$.

## Scalar Multiplication

One quality of vectors is that they can be multiplied by a scalar. Multiplying a vector by a scalar results in another vector that has an increased magnitude and/or a negated direction.

**Example K:** Given the vector $\boldsymbol{v} = 5\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}}$, increase its magnitude by a factor of 5.

$5\boldsymbol{v} = 5(5\hat{\boldsymbol{\imath}} + 6\hat{\boldsymbol{\jmath}})$

Step One
Multiply the vector by the scalar value 5, then simplify by using the distributive property of multiplication

$\boxed{5\boldsymbol{v} = (25\hat{\boldsymbol{\imath}} + 30\hat{\boldsymbol{\jmath}})}$

Solution

**Example L:** Reverse the direction of $5\boldsymbol{v}$ without changing its magnitude.

$-5\boldsymbol{v} = -1(25\hat{\boldsymbol{\imath}} + 30\hat{\boldsymbol{\jmath}})$

Step One
Multiply the vector by the scalar value -1, then simplify.

$\boxed{-5\boldsymbol{v} = (-25\hat{\boldsymbol{\imath}} - 30\hat{\boldsymbol{\jmath}})}$

Solution

**The Dot Product**

The operations of vector addition and scalar multiplication result in vectors. By contrast, multiplying two vectors together using the *dot product* results in a scalar. The dot product is written mathematically as $\boldsymbol{U} \cdot \boldsymbol{V}$ (reads "$\boldsymbol{U}$ dot $\boldsymbol{V}$"), where $\boldsymbol{U}$ and $\boldsymbol{V}$ are both vectors.

The dot product can only be used to multiply two vectors; it cannot be used to multiply a scalar and a vector, or a scalar and a scalar.

The dot product has many uses, such as finding the angle between two vectors, the projection of one vector onto another, the amount of work a force does on moving an object, etc. This handout will only go over how to calculate the dot product, not how to apply the dot product do various problems.

There are 2 main formulas used to calculate the dot product of two vectors:

**Equation 1:** Where $\boldsymbol{U}$ and $\boldsymbol{V}$ are arbitrary vectors, and $\boldsymbol{U} = \boldsymbol{U_x} + \boldsymbol{U_y}$, and $\boldsymbol{V} = \boldsymbol{V_x} + \boldsymbol{V_y}$,

$$\boxed{\boldsymbol{U} \cdot \boldsymbol{V} = \boldsymbol{U_x} * \boldsymbol{V_x} + \boldsymbol{U_y} * \boldsymbol{V_y}}$$

**Equation 2:** Where $\theta$ is the angle between $\boldsymbol{U}$ and $\boldsymbol{V}$,

$$\boxed{\boldsymbol{U} \cdot \boldsymbol{V} = |\boldsymbol{U}| * |\boldsymbol{V}| * cos(\theta)}$$

To determine which equation to use, review the given information, then determine which equation is the simplest to use. If the components of two vectors are given, the first equation is preferred in calculating the dot product. Likewise, if the magnitude and direction of the two vectors are given, the second equation is preferred.

**Example M:** Given that $\boldsymbol{U} = 10\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}}$, and $\boldsymbol{V} = 4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}}$, find $\boldsymbol{U} \cdot \boldsymbol{V}$.

$$\boldsymbol{U} \cdot \boldsymbol{V} = \boldsymbol{U_x} * \boldsymbol{V_x} + \boldsymbol{U_y} * \boldsymbol{V_y}$$

<u>Step One</u>

Because the magnitude of the components of vectors $\boldsymbol{U}$ and $\boldsymbol{V}$ is the only information given, equation 1 used.

$$\boldsymbol{U} \cdot \boldsymbol{V} = (10)(4) + (2)(3)$$

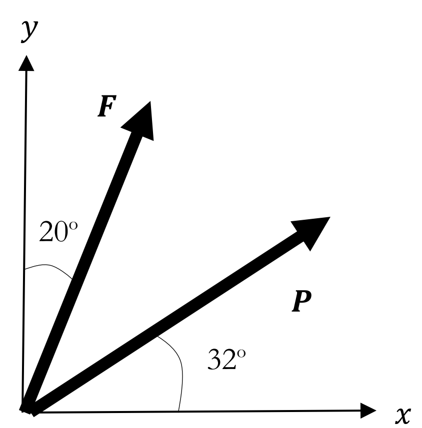
<u>Step Two</u>

Substitute the given information into the equation, then simplify.

$$\boxed{\boldsymbol{U} \cdot \boldsymbol{V} = 46}$$

<u>Solution</u>

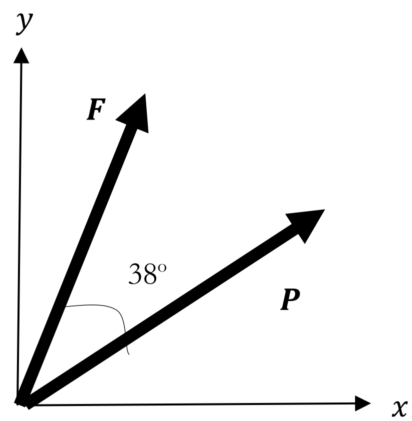
**Example N:** Given that $\boldsymbol{F}$ has a magnitude of 20N and a direction 20° to the right of the positive $y$-axis, and that $\boldsymbol{P}$ has a magnitude of 10N with a direction of 32° above the positive $x$-axis, find $\boldsymbol{F} \cdot \boldsymbol{P}$.




<u>Step One</u>

Begin by drawing the two vectors in the x-y plane

**Example N(continued):**




Step Two

Find that the angle between vectors **F** and **P** is equal to $90^o - (32^o + 20^o) = 38^o$

Step Three

Since the magnitude of both vectors, and the angle between the two vectors are known, use the second equation.

$$\boldsymbol{F} \cdot \boldsymbol{P} = |\boldsymbol{F}| * |\boldsymbol{P}| * cos(\theta)$$

Step Four

Substitute information into equation, then simplify.

$$\boldsymbol{F} \cdot \boldsymbol{P} = (20N)(10N) * cos(38^o)$$

Solution

$$\boxed{\boldsymbol{F} \cdot \boldsymbol{P} = 157.6}$$