Completing the Square

"Completing the square" is another method of solving quadratic equations. It allows trinomials to be factored into two identical factors.

Example: $x^2 + 4x + 4$

$$(x + 2)(x + 2) \text{ or } (x + 2)^2$$

To complete the square, it is necessary to find the constant term, or the last number that will enable factoring of the trinomial into two identical factors. To find the constant term needed, simply take the coefficient of "x," divide by 2, and square the quotient. If you have an equation, rather than an expression, the resulting number should be added to **both** sides of the equation.

Example: What is the constant term used to factor the expression $x^2 - 8x$ into two identical factors?

Step 1. Take the coefficient of "x", which is -8, and divide it by two.

$$\frac{-8}{2} = -4$$

Step 2. Take that number and square it.

$$(-4)^2 = 16$$

Step 3. Adding the constant term of 16 would allow the expression to be factored into identical factors.

$$x^2 - 8x + 16 = (x - 4)^2$$

To solve an equation by completing the square requires a couple of extra steps.

Example: Solve by completing the square $x^2 + 8x + 7 = 0$

Step 1. Move the constant term to the other side of the equation by subtracting from both sides.

$$x^2 + 8x + 7 - 7 = 0 - 7$$

$$x^2 + 8x = -7$$

Step 2. Complete the square.

$$\left(\frac{8}{2}\right)^2 = 4^2 = 16$$

Step 3. Since 16 is being added to the left side of the equation it **MUST** also be added to the right side.

$$x^2 + 8x + 16 = -7 + 16$$

$$x^2 + 8x + 16 = 9$$

Step 4. Factor the left side of the equation.

$$x^2 + 8x + 16 = 9$$

$$(x + 4)^2 = 9$$

HINT: the number inside the factor should always be the same as the number obtained from dividing the coefficient of "x" by two!

$\frac{8}{2} = 4$ and the factor was $(x + 4)^2$

Step 5. Take the square root of both sides and solve for x.

$$\sqrt[2]{(x + 4)^2} = \sqrt[2]{9}$$

$$x + 4 = \pm 3$$

$$x = -7 \text{ and } -1$$

To complete the square, the coefficient of x^2 must be one. If it is any other number, first divide the **entire equation** by that number.

Example: Solve by completing the square $4x^2 - 12x - 4 = 12$

Step 1. Divide the equation by 4 in order to get a leading coefficient of 1.

$$\frac{(4\mathrm{x}^2 - 12\mathrm{x} - 4)}{4} = \frac{12}{4}$$

$$x^2 - 3x - 1 = 3$$

Step 2. Move constants to the other side.

$$x^2 - 3x - 1 + 1 = 3 + 1$$

$$x^2 - 3x = 4$$

Step 3. Complete the square. Take the middle term, -3; divide by 2, and then square.

$$\left(\frac{-3}{2}\right)^2 = \frac{9}{4}$$

Step 4. Add that number to both sides of the equation.

$$x^2 - 3x + \frac{9}{4} = 4 + \frac{9}{4}$$

Find a common denominator between 4 and $\frac{9}{4}$ before adding them together. The common denominator is 4, so change 4 to $\frac{16}{4}$.

$$\frac{16}{4} + \frac{9}{4} = \frac{25}{4}$$

$$x^2 - 3x + \frac{9}{4} = \frac{25}{4}$$

Step 5. Factor the left side. Remember that the number inside the factor is the same as when you divide by two.

$$x^2 - 3x + \frac{9}{4} = \frac{25}{4}$$

$$\left(x - \frac{3}{2}\right)^2 = \frac{25}{4}$$

Step 6. Take the square root of both sides; solve for x.

$$\sqrt[2]{\left(x - \frac{3}{2}\right)^2} = \sqrt[2]{\frac{25}{4}}$$

$$x - \frac{3}{2} = \pm\frac{5}{2}$$

$x = -1$ and $x = 4$

Practice Problems

1. $x^2 + 6x + 5 = 0$

2. $x^2 + 8x - 9 = 0$

3. $x^2 - 6x + 9 = 0$

4. $x^2 + 4x - 7 = 0$

5. $x^2 - 5x - 24 = 0$

6. $x^2 - 8x + 15 = 0$

7. $4x^2 - 4x + 17 = 0$

8. $9x^2 - 12x + 13 = 0$

9. $4x^2 - 4x + 5 = 0$

10. $4x^2 - 8x + 1 = 0$

Answers to Practice Problems

1. -5 and -1
2. 1 and -9
3. 3 only
4. $-2-\sqrt{11}$ and $-2+\sqrt{11}$
5. -3 and 8
6. 3 and 5
7. $\frac{1}{2}+2i$ and $\frac{1}{2}-2i$
8. $\frac{2}{3}+i$ and $\frac{2}{3}-i$
9. $\frac{1}{2}+i$ and $\frac{1}{2}-i$
10. $1+\frac{\sqrt{3}}{2}$ and $1-\frac{\sqrt{3}}{2}$